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The end of the p-form hierarchy

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Abstract: The introduction of a non-abelian gauge group embedded into the rigid symmetry group G of a field theory with abelian vector fields and no corresponding charges, requires in general the presence of a hierarchy of p-form gauge fields. The full gauge algebra of this hierarchy can be defined independently of a specific theory and is encoded in the embedding tensor that determines the gauge group. When applied to specific Lagrangians, the algebra is deformed in an intricate way and in general will only close up to equations of motion. The group-theoretical structure of the hierarchy exhibits many interesting features, which have been studied starting from the low-p forms. Here the question is addressed what happens generically for high values of p. In addition a number of other features is discussed concerning the role that the p-forms play in various deformations of the theory.

Keywords: Duality in Gauge Field Theories, Gauge Symmetry, Supergravity Models, M-Theory.

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Contents

1. Introduction

In recent years the study of general gaugings of extended supergravities initiated in [1, 2] has led to considerable insight in the general question of embedding a non-abelian gauge group into the rigid symmetry group G of a theory that contains abelian vector fields without corresponding charges, transforming in some representation of G (usually not in the adjoint representation). The field content of this theory is fixed up to possible dualities between p-forms and $(d-p-2)$-forms. Therefore, it is advantageous to adopt a framework in which the decomposition of the form fields is determined only until after the gauging. The relevance of this can, for instance, be seen in four space-time dimensions [3], where the Lagrangian can be changed by electric/magnetic duality so that electric gauge fields are replaced by their magnetic duals. In the usual setting, one has to adopt an electric/magnetic duality frame where the gauge fields associated with the desired gauging are all electric. In principle this may not be sufficient, because the gauge fields should decompose under the embedded gauge group into fields transforming in the adjoint representation of the gauge group, and fields that are invariant under this group, so as to avoid inconsistencies. In a more covariant framework, on the other hand, one introduces both electric and magnetic gauge fields from the start, such that the desired gauge group can be embedded irrespectively of the particular electric/magnetic duality frame. Gauge charges can then be switched on in a fully covariant setting provided one introduces 2-form fields transforming in the adjoint representation of G. To keep the number of physical degrees of freedom unchanged, new gauge transformations associated with the 2-form gauge fields are necessary. In this approach the gauge group embedding is encoded in the so-called embedding tensor, which is treated as a spurionic quantity so as to make it amenable to group-theoretical methods.

This group-theoretical framework has already been applied to a rather large variety of supergravity theories in various space-time dimensions, where it was possible to characterize

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all possible gauge group embeddings in a group-theoretical fashion [4 − 10]. We note that the three-dimensional theories [11 − 13] also fall in this class, although they are special in that the vector gauge fields themselves can be avoided in the absence of any gauging, as they are dual to scalar fields. It is in this context that the embedding tensor was first introduced.

While in four space-time dimensions no p-form fields are required in the action beyond $p = 2$, the higher-dimensional case may incorporate higher-rank form fields which will naturally extend to a hierarchy when switching on gauge charges, inducing a non-trivial entanglement between forms of different ranks. It may seem that one introduces an infinite number of degrees of freedom in this way, but, as mentioned already above, the hierarchy contains additional gauge invariances beyond those associated with the vector fields. As it turns out, this hierarchy is entirely determined by the rigid symmetry group G and the embedding tensor that defines the gauge group embedding into G [2, 14] and a priori makes neither reference to an action nor to the number d of space-time dimensions. In particular, as a group-theoretical construct, the tensor hierarchy in principle continues indefinitely, but it can be consistently truncated in agreement with the space-time properties (notably the absence of forms of a rank $p > d$). In this paper we will analyze some of the generic features of the hierarchy for large values of the rank p (i.e. close to d).

Although every choice of embedding tensor defines a particular gauging and thereby a corresponding p-form hierarchy, it turns out that the hierarchy is universal in the sense that scanning through all possible choices of the embedding tensor and taking into account the group-theoretical representation constraints which it obeys, allows to characterize the multiplicity of the various p-forms in entire G-representations — within which every specific gauging selects its proper subset. This is precisely the meaning of treating the embedding tensor as a spurionic quantity. The covariant description of the gauged tensor hierarchy thereby enables the derivation of the full p-form field content. In the general case it may be difficult to indicate the precise G-representations to which the p-forms are assigned, but it is possible to indicate all the ingredients in a systematic way (although it requires some notational ingenuity) such that they can be worked out explicitly on a case-by-case basis [14].

Although the structure of the p-form hierarchy seems to be universal, the situation changes when incorporating this formalism in the context of a given Lagrangian. The transformation rules are then deformed by the presence of the various matter fields and, as a result, the closure of the generalized gauge algebra only holds up to equations of motion and additional symmetries (which are connected to certain redundancies in the transformation rules of the hierarchy) [6, 7, 14, 15]. Moreover the hierarchy is often truncated at a relatively early stage, because the Lagrangian may be such that the gauge transformations that connect to the higher-p forms have become trivially satisfied. This truncation process can be understood in the context of the hierarchy, because it can be truncated (at some value of p) by projecting the p-forms with the embedding tensor. For instance, in five space-time dimensions, the gauged supergravity Lagrangians do not require the presence of p-form fields with $p > 2$, because the 2-forms appear in the Lagrangian only in a certain contraction with the embedding tensor that precludes the continuation to higher-p

gauge invariances. On the other hand, the $(d-1)$- and d-forms play a different role, as was suggested in [14] where this role was explicitly demonstrated for three-dimensional maximal supergravity. The results of this paper indicate that this role is in fact generic, so that, while the hierarchy may be truncated at some specific value of p, the $(d-1)$- and d-forms can always be included.

It is a possiblity that the truncation induced by the Lagrangian is such that all p-forms with $p > 1$ decouple. In that case the hierarchy will not offer any new insights. But it does offer a universal framework in which gaugings must take place, although the field content of the theory will ultimately determine how much of the hierarchal structure will be reflected in the final result. On the other hand, the universal features of the hierarchy are presumably the reason why the results of this approach overlap in a surprising way with the results obtained in an entirely different context. For a discussion of some of these results we refer to the literature (see, e.g. [16 − 21, 14]).

This paper is organized as follows. In section 2 the hierarchy of p-form tensor fields is introduced in a general context. Section 3 presents the representations of the p-form fields for the maximally extended supergravities to illustrate some of the results that can be obtained in the context of the hierarchy. Section 4 deals with the question what the generic representations are for the higher-rank p-forms and discusses the possible role played by the $(d-1)$- and d-form fields. In the final section some consequences for more general deformations of ungauged theories are pointed out.

2. The *p*-form hierarchy

The p-form hierarchy has already been discussed in a number of places, but for clarity we summarize some of its main features here. We assume a theory with abelian gauge fields $A_\mu{}^M$, that is invariant under a group G of rigid transformations. The gauge fields transform in a representation of that group.[1] The generators in this representation are denoted by $(t_\alpha)_M{}^N$, so that $\delta A_\mu{}^M = -\Lambda^\alpha (t_\alpha)_N{}^M A_\mu{}^N$, and the structure constants $f_{\alpha\beta}{}^\gamma$ of G are defined according to $[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma\, t_\gamma$. The next step is to select a subgroup of G that will be elevated to a gauge group with non-trivial gauge charges, whose dimension is obviously restricted by the number of vector fields. The discussion in this section will remain rather general and will neither depend on G nor on the space-time dimension. We refer to [1, 4, 6] where a number of results was described for maximal supergravity in various dimensions.

The gauge group embedding is defined by specifying its generators X_M,[2] which couple to the gauge fields $A_\mu{}^M$ in the usual fashion, and which can be decomposed in terms of the independent G-generators t_α, i.e.,

$$X_M = \Theta_M{}^\alpha\, t_\alpha\,. \tag{2.1}$$

[1] In even space-time dimensions this assignment may fail and complete G representations may require the presence of magnetic duals. For four space-time dimensions, this has been demonstrated in [3].

[2] The corresponding gauge algebra may have a central extension acting exclusively on the vector fields.

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where $\Theta_M{}^\alpha$ is the *embedding tensor* transforming according to the product of the representation conjugate to the representation in which the gauge fields transform and the adjoint representation of G. This product representation is reducible and decomposes into a number of irreducible representations. Only a subset of these representations is allowed. For supergravity the precise constraints follow from the requirement of supersymmetry, but, from all applications worked out so far, we know that at least part (if not all) of the representation constraints is necessary for purely bosonic reasons such as gauge invariance of the action and consistency of the tensor gauge algebra. This constraint on the embedding tensor is known as the *representation constraint*. Here we treat the embedding tensor as a spurionic object, which we allow to transform under G, so that the Lagrangian and transformation rules remain formally G-invariant. At the end we will freeze the embedding tensor to a constant, so that the G-invariance will be broken. As was shown in [14] this last step can also be described in terms of a new action in which the freezing of $\Theta_M{}^\alpha$ will be the result of a more dynamical process.

The embedding tensor must satisfy a second constraint, the so-called *closure constraint*, which is quadratic in $\Theta_M{}^\alpha$ and more generic. This constraint ensures that the gauge transformations form a group so that the generators (2.1) will close under commutation. Any embedding tensor that satisfies the closure constraint, together with the representation constraint mentioned earlier, defines a consistent gauging. The closure constraint reads as follows,

$$\mathcal{Q}_{PM}{}^\alpha = \Theta_P{}^\beta t_{\beta M}{}^N \Theta_N{}^\alpha + \Theta_P{}^\beta f_{\beta\gamma}{}^\alpha \Theta_M{}^\gamma = 0\,, \tag{2.2}$$

and can be interpreted as the condition that the embedding tensor should be invariant under the embedded gauge group. Hence we can write the closure constraint as,

$$\mathcal{Q}_{MN}{}^\alpha \equiv \delta_M \Theta_N{}^\alpha = \Theta_M{}^\beta\, \delta_\beta \Theta_N{}^\alpha = 0\,, \tag{2.3}$$

where δ_M and δ_α denote the effect of an infinitesimal gauge transformation or an infinitesimal G-transformation, respectively. Contracting (2.2) with t_α leads to,

$$[X_M, X_N] = -X_{MN}{}^P\, X_P = -X_{[MN]}{}^P\, X_P\,. \tag{2.4}$$

It is noteworthy here that the generator $X_{MN}{}^P$ and the structure constants of the gauge group are related, but do not have to be identical. In particular $X_{MN}{}^P$ is in general not antisymmetric in $[MN]$. The embedding tensor acts as a projector, and only in the projected subspace the matrix $X_{MN}{}^P$ is antisymmetric in $[MN]$ and the Jacobi identity will be satisfied. Therefore (2.4) implies in particular that $X_{(MN)}{}^P$ must vanish when contracted with the embedding tensor. Denoting

$$Z^P{}_{MN} \equiv X_{(MN)}{}^P\,, \tag{2.5}$$

this condition reads,

$$\Theta_P{}^\alpha\, Z^P{}_{MN} = 0\,. \tag{2.6}$$

The tensor $Z^P{}_{MN}$ is constructed by contraction of the embedding tensor with G-invariant tensors and therefore transforms in the same representation as $\Theta_M{}^\alpha$ — except when the

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embedding tensor transforms reducibly so that $Z^P{}_{MN}$ may actually depend on a smaller representation. The closure constraint (2.3) then ensures that $Z^P{}_{MN}$ is gauge invariant. As is to be expected $Z^P{}_{MN}$ characterizes the lack of closure of the generators X_M. This can be seen, for instance, by calculating the direct analogue of the Jacobi identity,

$$X_{[NP}{}^R X_{Q]R}{}^M = \tfrac{2}{3} Z^M{}_{R[N} X_{PQ]}{}^R . \tag{2.7}$$

The fact that the right-hand side does not vanish has direct implications for the non-abelian field strengths: the standard expression

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{[NP]}{}^M A_\mu{}^N A_\nu{}^P , \tag{2.8}$$

which appears in the commutator $[D_\mu, D_\nu] = -g\mathcal{F}_{\mu\nu}{}^M X_M$ of covariant derivatives

$$D_\mu \equiv \partial_\mu - g\, A_\mu{}^M X_M , \tag{2.9}$$

is not fully covariant. Rather, under standard gauge transformations

$$\delta A_\mu{}^M = D_\mu \Lambda^M = \partial_\mu \Lambda^M + g A_\mu{}^N X_{NP}{}^M \Lambda^P , \tag{2.10}$$

the field strength $\mathcal{F}_{\mu\nu}{}^M$ transforms as

$$\begin{aligned} \delta\mathcal{F}_{\mu\nu}{}^M &= 2\, D_{[\mu}\delta A_{\nu]}{}^M - 2g\, X^M{}_{(PQ)}\, A_{[\mu}{}^P\, \delta A_{\nu]}{}^Q \\ &= g\, \Lambda^P X_{NP}{}^M\, \mathcal{F}_{\mu\nu}{}^N - 2g\, Z^M{}_{PQ}\, A_{[\mu}{}^P\, \delta A_{\nu]}{}^Q . \end{aligned} \tag{2.11}$$

This expression is *not* covariant — not only because of the presence of the second term on the right-hand side, but also because the lack of antisymmetry of $X_{NP}{}^M$ prevents us from obtaining the expected result by inverting the order of indices NP in the first term on the right-hand side. As a consequence, we cannot use $\mathcal{F}_{\mu\nu}{}^M$ in the Lagrangian. In particular, one needs suitable covariant field strengths for the invariant kinetic term of the gauge fields.

To remedy this lack of covariance, the strategy followed in [1, 2] has been to introduce additional (shift) gauge transformations on the vector fields,

$$\delta A_\mu{}^M = D_\mu \Lambda^M - g\, Z^M{}_{NP}\, \Xi_\mu{}^{NP} , \tag{2.12}$$

where the transformations proportional to $\Xi_\mu{}^{NP}$ enable one to gauge away those vector fields that are in the sector of the gauge generators $X_{MN}{}^P$ in which the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor by (2.6)). Fully covariant field strengths can then be defined upon introducing 2-form tensor fields $B_{\mu\nu}{}^{NP}$ belonging to the same representation as $\Xi_\mu{}^{NP}$,

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + g\, Z^M{}_{NP}\, B_{\mu\nu}{}^{NP} . \tag{2.13}$$

These tensors transform covariantly under gauge transformations

$$\delta\, \mathcal{H}_{\mu\nu}{}^M = -g \Lambda^P X_{PN}{}^M \mathcal{H}_{\mu\nu}{}^N , \tag{2.14}$$

provided we impose the following transformation laws for the 2-forms

$$Z^M{}_{NP}\,\delta B_{\mu\nu}{}^{NP} \;=\; Z^M{}_{NP}\left(2\,D_{[\mu}\Xi_{\nu]}{}^{NP} - 2\,\Lambda^N\mathcal{H}_{\mu\nu}{}^P + 2\,A_{[\mu}{}^N\delta A_{\nu]}{}^P\right). \tag{2.15}$$

We note that the constraint (2.6) ensures that

$$[D_\mu\,, D_\nu] \;=\; -\,g\mathcal{F}_{\mu\nu}{}^M X_M \;=\; -\,g\mathcal{H}_{\mu\nu}{}^M X_M\,, \tag{2.16}$$

but in the Lagrangian the difference between $\mathcal{F}^M$ and $\mathcal{H}^M$ is important.

Consistency of the gauge algebra thus requires the introduction of 2-form tensor fields $B_{\mu\nu}{}^{PN}$. It is important that their appearance in (2.13) strongly restricts their possible representation content. Not only must they transform in the symmetric product (NP) of the vector field representation as is manifest from their index structure, but also they appear under contraction with the tensor $Z^M{}_{NP}$ which in general does not map onto the full symmetric tensor product in its lower indices, but rather only on a restricted sub-representation. It is this sub-representation of G to which the 2-forms are assigned, and to keep the notation transparent, we denote the corresponding projector with special brackets $\lceil NP \rfloor$, such that

$$Z^M{}_{NP}\,B_{\mu\nu}{}^{NP} \;=\; Z^M{}_{NP}\,B_{\mu\nu}{}^{\lceil NP\rfloor}\,, \qquad \text{etc.}\,. \tag{2.17}$$

The tensor $Z^M{}_{NP}$ thus plays the role of an intertwiner between vector fields and 2-forms, which encodes the precise field content of the 2-form tensor fields such that the consistency of the vector gauge algebra is ensured.

The same pattern continues upon definition of a covariant field strength for the 2-forms and leads to a hierarchy of p-form tensor fields, which is entirely determined by choice of the global symmetry group G and its fundamental representation $\mathcal{R}_{\rm v}$ in which the vector fields transform. Let us collect its main features which have emerged in the study of particular gaugings and have been analyzed systematically in [2, 14] :

- Under the global symmetry group G of the theory, the p-forms transform in a sub-representation of the p-fold tensor product $\mathcal{R}_{\rm v}^{\otimes p}$, where $\mathcal{R}_{\rm v}$ denotes the representation of G in which the vector fields transform.In many cases of interest this is the fundamental representation. We denote these fields by

$$\overset{[1]}{A}{}^{M}\,,\quad \overset{[2]}{B}{}^{\lceil MN\rfloor}\,,\quad \overset{[3]}{C}{}^{\lceil M\lceil NP\rfloor\rfloor}\,,\quad \overset{[4]}{C}{}^{\lceil M\lceil N\lceil PQ\rfloor\rfloor\rfloor}\,,\quad \overset{[5]}{C}{}^{\lceil M\lceil N\lceil P\lceil QR\rfloor\cdot\cdot\rfloor}\,,\ \text{etc.}\,, \tag{2.18}$$

  where we have suppressed space-time indices, and the special brackets $\lceil\cdots\rfloor$ are introduced to denote the relevant sub-representations of $\mathcal{R}_{\rm v}^{\otimes p}$.

- The precise representation content of the $(p+1)$-forms $C_{[p+1]}{}^{\lceil N_0\lceil N_1\lceil\cdots N_p\rfloor\cdot\cdot\rfloor}$ are reflected in the *intertwining tensors* Y, defined recursively in terms of the lower-rank intertwiners and the gauge group generators X_{N_0} evaluated in the representation of the p-forms. For $p\geq 3$, this recursive relation is given by

$$\begin{aligned} Y^{M_1\lceil M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor}{}_{N_0\lceil N_1\lceil\cdots N_p\rfloor\cdot\cdot\rfloor} \equiv\; & -\,\delta_{N_0}^{\lceil M_1}\;Y^{M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor}{}_{N_1\lceil N_2\lceil\cdots N_p\rfloor\cdot\cdot\rfloor} \\ & -\,(X_{N_0})_{\lceil N_1\lceil N_2\lceil\cdots N_p\rfloor\cdot\cdot\rfloor}{}^{\lceil M_1\lceil M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor}\;. \end{aligned} \tag{2.19}$$

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Inspection of (2.19) for a concrete choice of G and $\mathcal{R}_{\rm v}$ shows that the intertwining tensor, considered as a map

$$Y^{[p]} : \mathcal{R}_{\rm v}^{\otimes(p+1)} \longrightarrow \mathcal{R}_{\rm v}^{\otimes p} \,, \tag{2.20}$$

has a non-trivial kernel whose complement defines the representation content of the $(p+1)$-forms that is required for consistency of the deformed p-form gauge algebra.

It is important to stress that all intertwining tensors depend linearly on the embedding tensor Θ. Since they are constructed from the embedding tensor contracted with G-invariant tensors, they all transform covariantly and belong to the same representation as the embedding tensor, in spite of their different index structure. Obviously the intertwining tensors depend on the particular gauging considered. However, sweeping out the full space of possible embedding tensors yields a Θ-independent (and G-covariant) result for the representation of $(p+1)$-forms. This is understood by regarding the embedding tensor as a so-called *spurionic* quantity, which transforms under the action of G, although at the end it will be fixed to a constant value. This approach shows how the mere consistency of the deformation of the p-form gauge algebra upon generic gaugings imposes rather strong restrictions on the field content of the ungauged theory. In the ungauged theory there is a priori no direct evidence for these restrictions and usually additional structures, such as supersymmetry or the underlying higher-rank Kac-Moody symmetries, motivate the presence and precise field content of the p-forms. It is rather surprising and intriguing that the constraints implied by these additional structures on the field content do precisely coincide with the constraints derived from consistency of the p-form hierarchy.

- The lowest-rank intertwining tensors are given by

$$Y^{[0]} : \mathcal{R}_{\rm v} \longrightarrow \mathcal{R}_{\rm adj} \,, \qquad Y^{[1]} : \mathcal{R}_{\rm v}^{\otimes 2} \longrightarrow \mathcal{R}_{\rm v} \,, \tag{2.21}$$

corresponding to $p = 0, 1$, with $(Y^{[0]})^{\alpha}{}_M = \Theta_M{}^{\alpha}$ and $(Y^{[1]})^M{}_{PQ} = Z^M{}_{PQ}$. For $p = 2$, the intertwining tensor can be written as follows,

$$Y^{MN}{}_{P\lceil RS\rfloor} = 2\,\delta_P{}^{\lceil M}\,Z^{N\rfloor}{}_{RS} - X_{P\lceil RS\rfloor}{}^{\lceil MN\rfloor} \,. \tag{2.22}$$

- Inspection of the symmetry properties of the intertwining tensors (2.21) and (2.19) shows explicitly that in general the lowest-rank p-forms in the hierarchy do not live in the full tensor product $\mathcal{R}_{\rm v}^{\otimes p}$, but only in a subsector thereof constrained by certain symmetry properties:

$$\overset{[1]}{A} \in \ydiagram{1}\,, \qquad \overset{[2]}{B} \in \ydiagram{2}\,, \qquad \overset{[3]}{C} \in \ydiagram{2,1}\,, \qquad \overset{[4]}{C} \in \ydiagram{3,1} \oplus \ydiagram{2,1,1}\,, \quad \text{etc.}\,, \tag{2.23}$$

in standard Young tableau notation.[3] In general, the group G will be different from an SL(N), so that the Young tableaux themselves are reducible. As it turns out, the tensor hierarchy then imposes further restrictions on the representation content.

[3]We should stress that the Young box '$\square$' here corresponds to the representation $\mathcal{R}_{\rm v}$ in which the vector fields transform under G and *not* to their space-time structure. With respect to the latter, all tensors of course transform as p-forms, i.e. in the totally antisymmetric part of the p-fold tensor product.

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- Mutual orthogonality: the intertwining tensors satisfy the relations

$$Y^{K_2\lceil K_3\lceil\cdots K_p\rfloor\cdot\cdot\rfloor}{}_{M_1\lceil M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor}\, Y^{M_1\lceil M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor}{}_{N_0\lceil N_1\lceil\cdots N_p\rfloor\cdot\cdot\rfloor} \approx 0\,, \tag{2.24}$$

  where 'weakly zero' (≈ 0) indicates that the expression vanishes as a consequence of the quadratic constraint (2.2) on the embedding tensor. More schematically, these orthogonality relations take the form

$$Y^{[p]}\cdot Y^{[p+1]} \approx 0\,, \tag{2.25}$$

  (with equation (2.6) as their lowest member) and thus in view of (2.20) define the sequence

$$\cdots \xrightarrow{Y^{[p+1]}} \mathcal{R}_{\rm v}^{\otimes(p+1)} \xrightarrow{Y^{[p]}} \mathcal{R}_{\rm v}^{\otimes p} \xrightarrow{Y^{[p-1]}} \quad \cdots \quad \xrightarrow{Y^{[1]}} \mathcal{R}_{\rm v} \xrightarrow{Y^{[0]}} \mathcal{R}_{\rm adj}\,. \tag{2.26}$$

  Again, we emphasize that every embedding tensor, i.e. every solution to the quadratic constraint, gives rise to such a sequence and defines its proper field content, while by sweeping out the entire space of possible embedding tensors one obtains the full p-form field content induced by the group G.

- Consequently, given the Y-tensors, and specifying the group G, the above results enable a complete determination of the full hierarchy of the higher-rank p-forms required for the consistency of the gauging. In particular, we can exhibit some of the terms in the variations of the p-form fields

$$\begin{aligned}\delta \overset{[p]}{C}{}^{M_1\lceil M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor} &= p\,\mathrm{D}\overset{[p-1]}{\Phi}{}^{M_1\lceil M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor} \\ &\quad + \Lambda^{\lceil M_1}\,\overset{[p]}{\mathcal{H}}{}^{\lceil M_2\cdots\rfloor\cdot\cdot\rfloor} + p\,\delta\overset{[1]}{A}{}^{\lceil M_1}\wedge\overset{[p-1]}{C}{}^{\lceil M_2\cdots\rfloor\cdot\cdot\rfloor} \\ &\quad - g\,Y^{M_1\lceil M_2\lceil\cdots M_p\rfloor\cdot\cdot\rfloor}{}_{N_0\lceil N_1\lceil\ldots N_p\rfloor.\rfloor}\,\overset{[p]}{\Phi}{}^{N_0\lceil N_1\lceil\ldots N_p\rfloor\cdot\cdot\rfloor}\,, \\ &\quad + \cdots\,. \end{aligned} \tag{2.27}$$

  In particular, this demonstrates how the intertwining tensors Y show up explicitly in the tensor gauge transformations to induce a Stückelberg-type coupling between p and $(p+1)$-forms. The dots in (2.27) represent further terms carrying the lower-rank p-forms such as terms linear in the covariant field strengths $\mathcal{H}$ (to be introduced below) and further Chern-Simons-like variations such as $\delta C\wedge C$.

- For all higher-rank p-forms covariant field strengths can be defined that transform homogeneously under vector gauge transformations and are invariant under all higher-rank tensor gauge transformations.
  E.g. for the 2-forms the modified field strength takes the form,

$$\begin{aligned}\overset{[3]}{\mathcal{H}}{}^{MN} &\equiv 3\,\mathrm{D}\overset{[2]}{B}{}^{MN} + 3\,\overset{[1]}{A}{}^{\lceil M}\wedge\left(\mathrm{d}\overset{[1]}{A}{}^{N\rfloor} + \tfrac{2}{3}gX_{[PQ]}{}^{N\rfloor}\,\overset{[1]}{A}{}^{P}\wedge\overset{[1]}{A}{}^{Q}\right) \\ &\quad + g\,Y^{MN}{}_{P\lceil RS\rfloor}\,\overset{[3]}{C}{}^{P\lceil RS\rfloor}\,. \end{aligned} \tag{2.28}$$

  This pattern continues.

- The hierarchy can be truncated at any value of p by projecting the corresponding forms with the next intertwining tensor. Because of the orthogonality property (2.24), the Stückelberg-type shifts are then no longer effective and the hierarchy will not be continued to higher p-forms. Of course, this projection is a somewhat arbitrary and technical way to truncate, but in practice this situation may occur when considering specific Lagrangians in which intertwining tensors may appear that effect precisely this projection. For instance, in five-dimensional maximal supergravity, the 3-form fields do not appear in the Lagrangian for precisely this reason.

Although the number of space-time dimensions does not enter into this analysis (as stated earlier, the iteration procedure can in principle be continued indefinitely), there exists, for the maximal supergravities, a consistent correlation between the rank of the tensor fields and the occurrence of conjugate G-representations that is precisely in accord with tensor-tensor and vector-tensor (Hodge) duality[4] corresponding to the space-time dimension where the maximal supergravity with that particular duality group G lives. In the next section we discuss some of the results of this analysis.

3. Representation assignments of the p-forms

The hierarchy of vector and tensor gauge fields that we presented in the previous section can be considered in the context of the maximal gauged supergravities. In that case the gauge group is embedded in the duality group G, which is known for each space-time dimension in which the supergravity is defined. Once the group G is specified, the hierarchy allows in principle a unique determination of the representations of the higher p-forms. Table 1 shows an overview of some of the results. We recall that the analysis described in section 2 did not depend on the number of space-time dimensions. For instance, it is possible to derive the representation assignments for (d+1)-rank tensors, although these do not live in a d-dimensional space-time (nevertheless, a glimpse of their existence occurs in d dimensions via the shift transformations (2.27) of the d-forms in the general gauged theory).

On the other hand, whenever there exists a (Hodge) duality relation between fields of different rank at the appropriate value for d, then one finds that their G representations turn out to be related by conjugation. This property is clearly exhibited at the level of the lower-rank fields in the table. More precisely, upon working out the precise representation content as described in the previous section, the sequence (2.26) takes the particular form

$$\cdots \xrightarrow{Y^{[d-2]}} \mathcal{R}_{\mathrm{adj}} \xrightarrow{Y^{[d-3]}} \mathcal{R}_{\mathrm{v}*} \xrightarrow{Y^{[d-4]}} \quad \cdots \quad \xrightarrow{Y^{[1]}} \mathcal{R}_{\mathrm{v}} \xrightarrow{Y^{[0]}} \mathcal{R}_{\mathrm{adj}}\ , \tag{3.1}$$

symmetric around the forms of rank $p = \frac{1}{2}[d-1]$, i.e. $\mathcal{R}_{\mathrm{v}*}$ denotes the representation dual to $\mathcal{R}_{\mathrm{v}}$, etc. . . In particular, the intertwiners in (3.1) are pairwise related by transposition

$$Y^{[0]} = (Y^{[d-3]})^{\mathrm{T}}\ , \qquad Y^{[1]} = (Y^{[d-4]})^{\mathrm{T}}\ , \qquad \text{etc.}\ . \tag{3.2}$$

It is intriguing that the purely group theoretical hierarchy reproduces the correct assignments consistent with Hodge duality. In particular, the assignment of the (d–2)-forms is in

[4] As well as with the count of physical degrees of freedom.

		1	2	3	4	5	6
7	SL(5)	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	SO(5,5)	$\mathbf{16}_c$	$\mathbf{10}$	$\mathbf{16}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$\mathrm{E}_{6(6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\mathbf{351}$	$\mathbf{27}+\mathbf{1728}$	
4	$\mathrm{E}_{7(7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{133}+\mathbf{8645}$		
3	$\mathrm{E}_{8(8)}$	$\mathbf{248}$	$\mathbf{1}+\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

Table 1: Duality representations of the vector and tensor gauge fields for gauged maximal supergravities in space-time dimensions $3 \leq d \leq 7$. The first two columns list the space-time dimension and the corresponding duality group.

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line with tensor-scalar duality, as these forms are dual to the Noether currents associated with the G symmetry. In this sense, the duality group G implicitly carries information about the space-time dimension.

What is more, the hierarchy naturally extends beyond the $(d-2)$-forms and thus to those non-propagating forms whose field content is not restricted by Hodge duality. It is another striking feature of the hierarchy that the diagonals pertaining to the $(d$–$1)$- and d-rank tensor fields refer to the representations conjugate to those assigned to the embedding tensor and its quadratic constraint, respectively. In the next section, we will show that this pattern is in fact generic and related to the special role these forms may play in the Lagrangian [14].

It is an obvious question whether these systematic features have a natural explanation in terms of M-theory and we refer to [14] for a discussion. Here it suffices to mention that the representation content agrees with results based on matrix models in M-theory [16], (see also, [22] and references quoted therein) where matrix theory [23, 24] is considered in a toroidal compactification. The representations in the table were also found in [17], where a 'mysterious duality' was exhibited between toroidal compactifications of M-theory and del Pezzo surfaces. Here the M-theory dualities are related to global diffeomorphisms that preserve the canonical class of the del Pezzo surface. Again the representations thus found are in good agreement with the representations in table 1. Furthermore there are hints that the above considerations concerning new M-theoretic degrees of freedom can be extended to infinite-dimensional duality groups. Already some time ago [18] it was shown from an analysis of the indefinite Kac-Moody algebra E_{11} that the decomposition of its so-called L1 representation at low levels under its finite-dimensional subalgebra $\mathrm{SL}(3) \times \mathrm{E}_8$ yields the same **3875** representation that appears for the 2-forms as shown in table 1. This analysis has meanwhile been extended [19 – 21] to other space-time dimensions and higher-rank forms, and again there is a clear overlap with the representations in table 1. Non-maximal supergravities have also been discussed from this perspective in [25, 26].

4. Life at the end of the hierarchy

Historically the p-form hierarchy was discovered by starting from the 1-forms belonging to the representation $\mathcal{R}_{\mathrm{v}}$, in the context of specific (supergravity) theories. The crucial ingredients are the group G and the representation of the embedding tensor. No information about the space-time dimension is required. On the other hand, one of the intitial observations was that general gaugings require a certain decomposition between certain p-forms and their duals, which belong to the conjugate representation. The actual distribution of physical degrees of freedom over these sets of fields related by duality is eventually determined by the value taken by the embedding tensor.

In this section, we will study the generic representation content of the p-forms predicted by the hierarchy for large rank p close to d. In view of the fact that the theory is invariant under the group G prior to switching on the gauge couplings, there exists a set of conserved 1-forms given by the Noether currents, transforming in the adjoint representation, which is dual to the (d–2)-forms. Furthermore we expect (d–3)-forms that are dual to the vector fields and thus are expected to transform in the G representation $\mathcal{R}_{\mathrm{v}*}$ dual to the vector field representation, in accordance with (3.1). When considering these high-rank p-forms it is convenient to switch from the general notation that was used in section 2 to a notation adapted to this particular field content and to identify the (d−3)- and (d−2)-forms as,

$$
\begin{aligned}
\overset{[d-3]}{C}{}^{M_1\lceil M_2\lceil\cdots M_{d-3}\rfloor\cdot\cdot\rfloor} &\sim \overset{[d-3]}{C}{}_{M}\,,\\
\overset{[d-2]}{C}{}^{M_1\lceil M_2\lceil\cdots M_{d-2}\rfloor\cdot\cdot\rfloor} &\sim \overset{[d-2]}{C}{}_{\alpha}\,,
\end{aligned}
\tag{4.1}
$$

upon explicit introduction of corresponding projectors, denoted by $\mathbb{P}^{M_1\lceil M_2\lceil\cdots M_{d-3}\rfloor\cdot\cdot\rfloor}{}_M$ and $\mathbb{P}^{M_1\lceil M_2\lceil\cdots M_{d-3}\rfloor\cdot\cdot\rfloor}{}_\alpha$. We may then explicitly study the end of the p-form hierarchy by imposing the general structure outlined in section 2. The result takes the following form,

$$
\begin{aligned}
\delta\,\overset{[d-3]}{C}{}_{M} &= (d-3)\,\mathrm{D}\,\overset{[d-4]}{\Phi}{}_{M} + \cdots - Y_M{}^{\alpha}\,\overset{[d-3]}{\Phi}{}_{\alpha}\,,\\
\delta\,\overset{[d-2]}{C}{}_{\alpha} &= (d-2)\,\mathrm{D}\,\overset{[d-3]}{\Phi}{}_{\alpha} + \cdots - Y_{\alpha,M}{}^{\beta}\,\overset{[d-2]}{\Phi}{}^{M}{}_{\beta}\,,\\
\delta\,\overset{[d-1]}{C}{}^{M}{}_{\alpha} &= (d-1)\,\mathrm{D}\,\overset{[d-2]}{\Phi}{}^{M}{}_{\alpha} + \cdots - Y^{M}{}_{\alpha,PQ}{}^{\beta}\,\overset{[d-1]}{\Phi}{}^{PQ}{}_{\beta}\,,\\
\delta\,\overset{[d]}{C}{}^{MN}{}_{\alpha} &= d\,\mathrm{D}\,\overset{[d-1]}{\Phi}{}^{MN}{}_{\alpha} + \cdots - Y^{MN}{}_{\alpha,PQR}{}^{\beta}\,\overset{[d]}{\Phi}{}^{PQR}{}_{\beta}\,,\\
\delta\,\overset{[d+1]}{C}{}^{PQR}{}_{\alpha} &= (d+1)\,\mathrm{D}\,\overset{[d]}{\Phi}{}^{PQR}{}_{\alpha} + \cdots\,,
\end{aligned}
\tag{4.2}
$$

where we indicated the most conspicuous parts of the p-form transformations. We included the transformations associated to the (d+1)-form for reasons that will be explained shortly.

From the index structure it is obvious that $Y_M{}^{\alpha}$ must coincide with the embedding tensor. The subsequent intertwining tensors can then be found by applying (2.19) which yields[5]

$$
\begin{aligned}
Y_{\alpha,M}{}^{\beta} &= t_{\alpha M}{}^{N}\, Y_N{}^{\beta} - X_{M\alpha}{}^{\beta}\ , \\
Y^M{}_{\alpha,PQ}{}^{\beta} &= -\,\delta_P{}^M\, Y_{\alpha,Q}{}^{\beta} - (X_P)_Q{}^{\beta,M}{}_{\alpha}\ , \\
Y^{MN}{}_{\alpha,PQR}{}^{\beta} &= -\,\delta_P^M\, Y^N{}_{\alpha,QR}{}^{\beta} - (X_P)_{QR}{}^{\beta,MN}{}_{\alpha}\ .
\end{aligned} \tag{4.3}
$$

The presence of the generator $t_{\alpha M}{}^{N}$ in the first equation is related to the conversion of the special bracket notation employed in the previous sections.

It is, however, more instructive to cast these expressions into a different form, given by

$$
\begin{aligned}
Y_M{}^{\alpha} &= \Theta_M{}^{\alpha}\ , \\
Y_{\alpha,M}{}^{\beta} &= \delta_\alpha \Theta_M{}^{\beta}\ , \\
Y^M{}_{\alpha,PQ}{}^{\beta} &= -\,\frac{\partial \mathcal{Q}_{PQ}{}^{\beta}}{\partial\, \Theta_M{}^{\alpha}}\ , \\
Y^{MN}{}_{\alpha,PQR}{}^{\beta} &= -\delta_P^M\, Y^N{}_{\alpha,QR}{}^{\beta} - X_{PQ}{}^{M}\, \delta_R^N \delta_\alpha^\beta - X_{PR}{}^{N}\, \delta_Q^M \delta_\alpha^\beta + X_{P\alpha}{}^{\beta}\, \delta_R^N \delta_Q^M\ ,
\end{aligned} \tag{4.4}
$$

where sign factors have been adopted such that the above tensors are precisely consistent with (4.3). In this form, it is straightforward to verify that the intertwining tensors satisfy the mutual orthogonality property (2.24). For the first few tensors this is easy to prove,

$$
\begin{aligned}
Y_M{}^{\alpha}\, Y_{\alpha,N}{}^{\beta} &= \delta_M \Theta_N{}^{\beta} \;=\; \mathcal{Q}_{MN}{}^{\beta} \;\approx\; 0\ , \\
Y_{\alpha,N}{}^{\beta}\, Y^N{}_{\beta,PQ}{}^{\gamma} &= \delta_\alpha\, \mathcal{Q}_{PQ}{}^{\gamma} \;\approx\; 0\ ,
\end{aligned} \tag{4.5}
$$

where we recall the constraint written as in (2.3). In the second equation we used the fact that the intertwining tensors are all G-covariant, so that the effect of transforming the embedding tensor is equivalent to transforming the tensor according to its index structure.

The last orthogonality relation is proved differently. First we note the identity,

$$
Y^{MN}{}_{\alpha,PQR}{}^{\beta}\; \mathcal{Q}_{MN}{}^{\alpha} = 0\ , \tag{4.6}
$$

which holds identically without making reference to the quadratic constraint (2.2). This is thus a non-trivial identity that is cubic in the embedding tensor. It follows by comparing

$$
\delta_P\, \mathcal{Q}_{QR}{}^{\beta} = \delta_P \Theta_N{}^{\alpha}\, \frac{\partial \mathcal{Q}_{QR}{}^{\beta}}{\partial\, \Theta_N{}^{\alpha}} = \Big(-\,\delta_P^M\, Y^N{}_{\alpha,QR}{}^{\beta}\Big)\, \mathcal{Q}_{MN}{}^{\alpha}\ , \tag{4.7}
$$

to

$$
\delta_P\, \mathcal{Q}_{QR}{}^{\beta} = \Big(X_{PQ}{}^{M}\, \delta_R^N \delta_\alpha^\beta + X_{PR}{}^{N}\, \delta_Q^M \delta_\alpha^\beta - X_{P\alpha}{}^{\beta}\, \delta_R^N \delta_Q^M\Big)\, \mathcal{Q}_{MN}{}^{\alpha}\ . \tag{4.8}
$$

[5]It is important to realize that (2.19) is only valid for $p \geq 3$, which implies that these results cannot be directly applied to low space-time dimensions. However, in that case the intertwining tensors are already known and given by (2.21) and (2.22).

This last equation follows from the fact that the tensor $\mathcal{Q}_{QR}{}^{\beta}$ transforms covariantly. Taking the difference of the two equations (4.7) and (4.8) leads directly to (4.6).

The importance of this result will be discussed below, but we first note that the missing orthogonality relation between the intertwiners follows from taking the derivative of (4.6) with respect to the embedding tensor,

$$\begin{aligned} Y^{M}{}_{\alpha,KL}{}^{\beta}\, Y^{KL}{}_{\beta,PQR}{}^{\gamma} &= -\,Y^{KL}{}_{\beta,PQR}{}^{\gamma}\,\frac{\partial \mathcal{Q}_{KL}{}^{\beta}}{\partial\,\Theta_M{}^{\alpha}} \\ &= \frac{\partial Y^{KL}{}_{\beta,PQR}{}^{\gamma}}{\partial\,\Theta_M{}^{\alpha}}\,\mathcal{Q}_{KL}{}^{\beta} \;\approx\; 0\,. \end{aligned} \tag{4.9}$$

From (4.4) we can now directly read off the representation content of the (d–1)- and the d-forms that follows from the hierarchy: the form of $Y_{\alpha,M}{}^{\beta}$ and $Y^{M}{}_{\alpha,PQ}{}^{\beta}$ shows that these forms transform in the representations dual to the embedding tensor $\Theta_M{}^{\beta}$ and the quadratic constraint $\mathcal{Q}_{PQ}{}^{\beta}$, respectively. As such, they can naturally be coupled, acting as Lagrange multipliers enforcing the property that the embedding tensor is space-time independent and gauge invariant [14]. This idea has been worked out explicitly in the context of maximal supergravity in three space-time dimensions, and we will demonstrate here that it can also be realized in a more general context. Hence we view the embeddding tensor as a space-time dependent scalar field, transforming in the G-representation constrained by possible representation constraints. To the original Lagrangian $\mathcal{L}_0$ which may depend on p-forms with $p \leq d-2$, we then add the following interactions,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm C}\,, \tag{4.10}$$

with

$$\mathcal{L}_{\rm C} \propto \varepsilon^{\mu_1\cdots\mu_d}\left\{d\,g\,C_{\mu_2\cdots\mu_d}{}^{M}{}_{\alpha}\,D_{\mu_1}\Theta_M{}^{\alpha} + g^2\,C_{\mu_1\cdots\mu_d}{}^{MN}{}_{\alpha}\,\mathcal{Q}_{MN}{}^{\alpha}\right\}\,, \tag{4.11}$$

where $\Theta_M{}^{\alpha}(x)$ is now a field. First we note that this Lagrangian is invariant under the shift transformation of the d-rank tensor field, by virtue of the identity (4.6). Varying this Lagrangian with respect to $\Theta_M{}^{\alpha}$ leads to the following variation,

$$\begin{aligned} \delta\mathcal{L}_{\rm C} \propto{}& -g\,\varepsilon^{\mu_1\cdots\mu_d}\,\delta\Theta_M{}^{\alpha} \\ &\times\Big[d\,D_{\mu_1}C_{\mu_2\cdots\mu_d}{}^{M}{}_{\alpha} + g\,Y^{M}{}_{\alpha,PQ}{}^{\beta}\,C_{\mu_1\cdots\mu_d}{}^{PQ}{}_{\beta} + d\,g\,A_{\mu_1}\,Y_{\alpha,N}{}^{\beta}\,C_{\mu_2\cdots\mu_d}{}^{N}{}_{\beta}\Big]\,. \end{aligned} \tag{4.12}$$

This result can be written as follows,

$$\delta\mathcal{L}_{\rm C} \propto -g\,\varepsilon^{\mu_1\cdots\mu_d}\left[\mathcal{H}_{\mu_1\cdots\mu_d}{}^{M}{}_{\alpha} + d\,gA_{[\mu_1}{}^{M}\,\mathcal{H}_{\mu_2\cdots\mu_d]\,\alpha} + \cdots\right]\delta\Theta_M{}^{\alpha}\,, \tag{4.13}$$

by including unspecified terms involving form fields of rank $p \leq d-2$. These terms are assumed to originate from the Θ-variation of the Lagrangian $\mathcal{L}_0$, but they cannot be evaluated in full generality as this depends on the details of the latter Lagrangian.

Qualitatively the above result is quite similar to that obtained in three space-time dimensions, but there are slight differences in the numerical factors, due to the fact that the three-dimensional result involves the intertwining tensors for low p-values, whereas the result here is based on generic $p \geq 3$ intertwining tensors. For Lagrangians that contain at most two derivatives, the Lagrangian will depend at most quadratically on $\Theta_M{}^\alpha(x)$. Hence this field may be integrated out, precisely as discussed in three space-time dimensions [14], so that all possible gaugings are comprised in one single Lagrangian.

5. Concluding remarks

The gaugings accompanied by a p-form hierarchy can be considered as a class of deformations of the original theory (which was invariant under the group G), induced by switching on certain charges. These charges necessarily generate a subgroup of G, extended by a variety of p-form gauge transformations. In principle, other deformations can be envisaged and one may wonder whether they can be switched on at the same time and/or whether they are completely independent.

An example constitutes the massive deformation known from IIA supergravity in ten dimensions [27], which is a priori unrelated to a gauging. However, let us reconsider the orthogonality relation (2.6),

$$\Theta_M{}^\alpha \, Z^M{}_{NP} \;=\; 0\,, \tag{5.1}$$

which can be trivially satified by setting $\Theta_M{}^\alpha = 0$. In view of the hierarchy (2.26), this deformation corresponds to a sequence in which the lowest map $Y^{[0]}$ is absent such that the hierarchy is not induced by the gauge interactions but starts at the level of the 2-forms. It would be interesting to analyze the general conditions under which such additional deformations can be launched from higher-ranks in the hierarchy, e.g. under which conditions the intertwining tensor Z can contain representations beyond those determined by the embedding tensor (2.5).

There is one other aspect that should be stressed. The gaugings are controlled by the coupling constant g, and one may consider taking the limit $g \to 0$. In that limit the covariant tensor hierarchy does not reduce to a trivial abelian set of tensor gauge fields but also reproduces non-trivial terms of order g^0. Consider as an example the covariant field strength $\mathcal{H}_{\mu\nu\rho}{}^{MN}$, defined in (2.28), which contains Chern-Simons-like terms that are not of order g. This feature, which may seem somewhat surprising, was first noted in five-dimensional maximal supergravity, where a Chern-Simons coupling is required by supersymmetry. However, this Chern-Simons coupling is a special case of the Chern-Simons coupling that is required by the gauge hierarchy [1]. To put it differently, if supersymmetry would have excluded the presence of a Chern-Simons coupling, then this theory could not have been deformed by gauge interactions.

Finally, let us mention that for groups G other than the series related to the maximal supergravities listed in table 1, the tensor hierarchy that we have exploited in this paper, may not run continuously all the way from scalar fields to d-forms, but break off at some earlier stage. This happens e.g. for the groups G = GL(n) and G = SO(n,n) for which the hierarchy breaks off (upon imposing a mild assumption regarding the representation constraints) after the vector and the 2-form fields, respectively. Accordingly, the associated theories are not linked to specific space-time dimensions — but correspond to the T^n

torus reduction of pure gravity and bosonic string theory, respectively, in an arbitrary dimension. The corresponding sequences (3.1) will thus exhibit an adequate gap in the middle, while the structure of forms with $p \geq (d-3)$ remains the generic one that we have discussed in section 4. Another example in which the hierarchy is degenerate concerns ten-dimensional IIB supergravity, which carries only forms of even degree such that (3.1) cannot be established.

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